JOURNAL OF ECONOMIC THEORY 5, 395–418 (1972)

Markets in Licenses and Efficient Pollution Control Programs*

W. DAVID MONTGOMERY

Division of the Humanities and Social Sciences, California Institute of Technology, Pasadena, California 91109

Received May 19, 1972

1. INTRODUCTION

Artificial markets have received some attention as a means of remedying market failure and, in particular, dealing with pollution from various sources. Arrow [1] has demonstrated that when externalities are present in a general equilibrium system, a suitable expansion of the commodity space would lead to Pareto optimality by bringing externalities under the control of the price system. Since his procedure is to define new commodities, each of which is identified by the type of externality, the person who produces it and the person who suffers it, his conclusion is pessimistic. Each market in the newly defined commodities involves but one buyer and one seller, and no forces exist to compel the behavior which would bring about a competitive equilibrium.

On the other hand, many forms of pollution are perfect substitutes for each other. Sulfur oxide emissions from one power plant trade off in the preferences of any sufferer with sulfur oxide emissions from some other power plant at a constant rate. This fact leads to the possibility of establishing markets in rights (or "licenses") which will bring together many buyers and sellers. Dales [2] has discussed a wide variety of such arrangements.

Unfortunately, because of the elements of public goods present in most environmental improvements, it appears unlikely that markets in rights, containing many sufferers from pollution as participants, will lead to overall Pareto optimality. They can only serve the more limited, but still

* Parts of this article appeared in my Ph.D. dissertation "Market Systems for the Control of Air Pollution," submitted to the Department of Economics at Harvard University in May, 1971. Research on this thesis was partly supported under Grant No. AP-00842 from the Environmental Protection Agency to Walter Isard. I am also indebted to Kenneth Arrow and James Quirk for valuable advice. Needless to say, all errors are solely the responsibility of the author.

valuable, function of achieving specified levels of environmental quality in an efficient manner. An example of this function is found in a proposal by Jacoby and Schaumburg [6] to establish a market in licenses (or "BOD bonds") to control water pollution from industrial sources in the Delaware estuary. The purpose of the present article is to provide a solid theoretical foundation for such proposals. Markets such as those proposed by Jacoby and Schaumburg will be characterized in a general fashion, and it will be proved that even in quite complex circumstances the market in licenses has an equilibrium which achieves externally given standards of environmental quality at least cost to the regulated industries.

Two types of license are discussed: a "pollution license," and an "emission license." The emission license directly confers a right to emit pollutants up to a certain rate. The pollution license confers the right to emit pollutants at a rate which will cause no more than a specified increase in the level of pollution at a certain point. Since a polluter will in general affect air or water quality at a number of points as a result of his emissions, he will be required to hold a portfolio of licenses covering all relevant monitoring points. All such licenses are free transferable. A main thesis of this article is that the market in pollution licenses will be more widely applicable than the market in emission licenses.

1.1. *The Applicable Pollution Control Problem*

Consider the following problem of pollution control: In a certain region there is a set of n industrial sources of pollution, each of which is fixed in location and owned by an independent, profit-maximizing firm. The prices of the inputs and outputs of these firms are fixed, because the region is small relative to the entire economy. Therefore any change in the level of output of a firm or industry in the region will have only a negligible impact on the output of the economy as a whole, and prices will be unaffected by output changes in the region. These firms are represented by a set of integers $I = \{1,..., n\}$.

Some regional standard of environmental quality in terms of a single pollutant has been chosen as a goal by a resource management agency. This standard is denoted by a vector $Q^* = (q_1^*,..., q_m^*)$. If air pollution is the particular area of interest, q_j^* might be an annual average concentration of sulfur dioxide at point j in an air basin. If water pollution is involved, q_j^* might be a measure of dissolved oxygen deficit at point j on a river. Since there is only one pollutant present in the region, the elements of Q^* represent concentrations of the one pollutant at various locations. The development of a decentralized system for achieving environmental goals at a number of different locations is the most important contribution of this article.

All pollution in the region arises from the industrial sources, each of which emits a single pollutant at the rate e_i. The emission vector $E = (e_1, ..., e_n)$ is mapped into concentrations by a semipositive matrix H, so that $E \cdot H = Q$. The standard Q^* imposes constraints on allowable emission rates of the form $E \cdot H \leqslant Q^*$. The problem of pollution control is to achieve Q^* at least cost to the polluters.

Some discussion of the limitations which the model places on the results presented is in order. The assumption that concentrations are a linear function of emissions is the only part of this problem which does not generalize easily. Therefore, the market in licenses to be described must be construed as applicable only in situations in which the assumption is approximately true. Fortunately, there are at least two important problems of pollution control in which it is true. One such is the management of dissolved oxygen deficit in a river. The DOD at a point downstream of a source releasing BOD (bacteriological oxygen demand) effluent is proportional to the BOD released [6, 8].

Management of concentrations of nonreactive atmospheric pollutants is another problem in which linearity is approximately true, as long as the variables to be related are average emission rates and average concentrations [5]. In this case, which will be used in this article as the source of illustrative examples, a meteorological diffusion model provides the means of relating long-run average concentrations to average rates of emission. As formulated by Martin and Tikvart [11], the model is based on an equation describing the shape of a smoke plume from an elevated source emitting at a constant rate with a wind of constant direction and speed. From this equation, the contribution of any source to concentration at any receptor can be calculated for given wind direction and speed. By taking the frequency distribution of wind direction and speed and appropriately modifying the predicted concentrations, one arrives at a theoretical relationship between average rates of emission and average concentrations [14].

The results of the diffusion model can be conveniently represented as an $m \times n$ matrix of unit diffusion coefficients, denoted

$$H = \begin{pmatrix} & \vdots & \\ \cdots & h_{ij} & \cdots \\ & \vdots & \end{pmatrix}.$$

The typical element states the contribution which one unit of emission by firm i makes to average pollutant concentration at point j.

The assumption that only one pollutant is present in the region can be justified by appeal to the external decision on desired air quality. If desired air quality in terms of one pollutant is independent of desired air quality in terms of any other so that, for example, the decision on the desirability of a certain concentration of sulfur dioxide is independent of the concentration of particulates permitted in the region, then nothing is lost. The management problem can be generalized by adding constraints representing emission vectors which achieve desired levels of many pollutants and joint production of pollution. In principle, it is solved in the same way as the one-pollutant system developed here.

The assumption that all prices (except those associated with pollution) are unaffected by measures undertaken to control pollution is a common one in economic analysis of environmental problems. It is necessary to allow consideration of problems in isolation, and to avoid full-sized (and nonoperational) general-equilibrium models [9].

When this assumption is made, it is possible to define for each firm a single-valued function which associates a cost with any emission rate adopted by the firm.

1.2. *The Cost Function*

The purpose of this section is to construct a function relating each level of emission which might be adopted by the firm to its cost and to establish that the profit-maximizing firm will minimize this function. Moreover, it will be argued that no firm will ever choose a level of emission greater than that which is observed in the complete absence of regulation.

Consider the typical multiproduct firm i. Let

$$G_i(y_{i1},..., y_{iR}, e_i)$$

represent the minimum total cost of producing a vector of output $(y_{i1},..., y_{iR})$ and emissions e_i. This is the cost incurred when inputs are optimally adjusted for that output and emission level. For the static analysis with which we deal, we can assume that both operating costs and an annual capital cost are included. Profit then will be

$$\pi_i = \sum_r p_r y_{ir} - G_i(y_{i1},..., y_{iR}, e_i).$$

Assume that G_i is convex and twice differentiable and that its domain is the positive orthant of the $r + 1$-dimensional space of real numbers. Define $(\bar{y}_{i1},..., \bar{y}_{ir}, \bar{e}_i)$ by

$$\sum_r p_r \bar{y}_{ir} - G_i(\bar{y}_{i1},..., \bar{y}_{iR}, \bar{e}_i) = \max_{y_{ir}, e_i} \left[\sum_r p_r y_{ir} - G_i(y_{i1},..., y_{iR}, e_i)\right].$$

Now consider the case in which the firm must adopt an emission level e_i and adjusts its output in order to obtain maximum profit for the fixed level of emission. Define $\tilde{y}_{ir}$ by

$$\sum_r p_r \tilde{y}_{ir} - G_i(\tilde{y}_{i1}, \dots, \tilde{y}_{iR}, e_i) = \max_{y_{ir}} \left[\sum_r p_r y_{ir} - G_i(y_{i1}, \dots, y_{iR}, e_i)\right].$$

The cost to firm i of adopting emission level e_i is defined as the difference between its unconstrained maximum of profit and its maximum of profit when emissions equal e_i. That is,

$$F_i(e_i) = \sum_r p_r(\bar{y}_{ir} - \tilde{y}_{ir}) - [G_i(\bar{y}_{i1}, \dots, \bar{y}_{iR}, \bar{e}_i) - G_i(\tilde{y}_{i1}, \dots, \tilde{y}_{iR}, e_i)]. \tag{1.1}$$

This cost is composed of two terms: the change in gross income from altering the output vector and the change in costs from setting emissions at a nonoptimal level (with an optimal adjustment of output).[1]

Consider the variation in $F_i(e_i)$ when a small change is made in e_i. Differentiating totally with respect to e_i, we find

$$dF_i(e_i) = -\sum_r \left(p_r - \frac{\partial G_i}{\partial y_{ir}}\right) \frac{dy_{ir}}{de_i} de_i + \frac{\partial G_i}{\partial e_i} de_i. \tag{1.2}$$

We have assumed that output levels adjust to maximize profit for a given level of e_i. That is, y_{ir} adjusts so that

$$p_r - \partial G_i / \partial y_{ir} = 0$$

for $j = 1, \dots, r$. Therefore [13],

$$dF_i(e_i)/de_i = \partial G_i / \partial e_i. \tag{1.3}$$

It can further be shown that the convexity of $G_i(y_{i1}, \dots, y_{iR}, e_i)$ implies the convexity of $F_i(e_i)$.

THEOREM 1.1. *If $G_i(y_{i1}, \dots, y_{iR}, e_i)$ is convex, $F_i(e_i)$ is also convex.*

Proof. The proof is immediate from the definition of convexity.

It is convenient to be able to use a single-valued function $F_i(e_i)$ to associate with any emissions level its cost. The properties of $F_i(e_i)$ proved above allow us to conclude that any relevant conditions which are satisfied

[1] Three general classes of techniques of emission reduction are available. First, emissions can be reduced by reducing the scale of output, or by altering the product mix of the firm. Second, the production process or the inputs used, such as fuels, can be altered. Finally, "tail-end" cleaning equipment can be installed to remove pollutants from effluent streams before they are released into the environment. All three of these techniques will commonly be found in combination.

by the partial derivative of G_i with respect to e_i will be satisfied by the derivative of F_i. In particular, we can conclude that if the profit-maximizing firm has any choice of e_i, it will minimize $F_i(e_i)$ subject to whatever costs or constraints we impose on it. Moreover, if G_i is convex, it follows that the conditions under which $\sum_i F_i(e_i)$ is minimized are the same as the conditions under which the total economic cost to firms of emissions control is minimized.

2. The Characterization of an Efficient Emission Vector

The goal of management is limited to bringing about an emission vector which will result in air of quality Q^* at least total cost to the region. Such an emission vector is called efficient, and designated E^{**}. The concept of least cost to the community is also given a specific meaning: it is the minimum of the sum $\sum_i F_i(e_i)$. With some risk of ambiguity, this sum is called "joint total cost."

To provide a reference to which later results can be compared, a general solution for the efficient emission vector can now be derived. The problem is to choose the vector $E = (e_1, \ldots, e_n)$ to minimize $\sum_i F_i(e_i)$ subject to the constraints

$$E \geqslant 0 \qquad \text{and} \qquad EH \leqslant Q^*,$$

where $Q^* \geqslant 0$, $h_{ij} \geqslant 0$ for all i, j. We will label this the "total joint cost minimum problem." Our exploration will proceed throughout this article on the assumption that G_i is convex. This implies that $F_i(e_i)$ is convex, and therefore that $\sum_i F_i(e_i)$ is also convex. It is also assumed that H is semipositive. The typical shape of $F_i(e_i)$ is illustrated in Fig. 1.

Minimizing a convex function subject to linear constraints and non-negativity constraints is equivalent to finding the saddle point of an associated Lagrangean. Formally, $(E^{**}, U^{**}) = (e_1^{**}, \ldots, e_n^{**}, u_1^{**}, \ldots, u_m^{**})$ will be a saddle point of the expression

$$-\sum_i F_i(e_i) + \sum_j u_j \left(q_j^* - \sum_i h_{ij} e_i\right)$$

with $E^{**} \geqslant 0$, $U^{**} \geqslant 0$. The differential Kuhn–Tucker conditions for this saddle point are

$$F_i'(e_i) + \sum_j u_j h_{ij} \geqslant 0, \qquad \sum_i \left[e_i \left(F_i'(e_i) + \sum_j u_j h_{ij}\right)\right] = 0, \qquad (2.1)$$

$$q_j^* - \sum_i h_{ij} e_i \geqslant 0, \qquad \sum_j \left[u_j \left(q_j - \sum_i h_{ij} e_i\right)\right] = 0. \qquad (2.2)$$

These conditions are necessary and sufficient [7]. Moreover, it is easy to show that the minimum does in fact exist.

THEOREM 2.1. *E^{**} and U^{**} satisfying* (2.1) *and* (2.2) *exist.*

Proof. Since $\sum_i F_i(\bar{e}_i) = 0$ and $\sum_i F_i(e_i) \geqslant \sum_i F_i(\bar{e}_i)$, for $e_i \geqslant 0$, $\sum_i F_i(e_i)$ is bounded from below. By hypothesis, the set

$$\Psi = \{E \mid EH \leqslant Q^*, E \geqslant 0\}$$

is not empty. Therefore, $\sum_i F_i(e_i)$ is defined on a nonempty closed set and bounded from below; therefore, it attains a minimum over the set Ψ for some element of Ψ.

If $\sum_i F_i(e_i)$ is not strictly convex, then E^{**} need not be unique. Since, however, $\min_{E\in\Psi} \sum_i F_i(e_i)$ is unique, it does not matter what particular minimizer is chosen. Therefore, I shall refer to the vector which minimizes costs; the reader may interpret this reference as meaning "any element of the set of E which minimizes $\sum_i F_i(e_i)$."

The following theorem is true if $\sum_i F_i(e_i)$ is strictly convex.

THEOREM 2.2. *If E^{**} minimizes $\sum_i F_i(e_i)$ subject to $EH \leqslant Q^*$ and $E \geqslant 0$, then $E^{**} \leqslant \bar{E}$.*

Proof. Assume *per contra* that $e_i^{**} > \bar{e}_i$ for some $i = i'$. Then $F_{i'}(e_{i'}^{**}) > F_{i'}(\bar{e}_{i'})$ and $h_{ij}e_{i'}^{**} > h_{ij}\bar{e}_{i'}$. Therefore

$$\sum_{i \neq i'} F_i(e_i^{**}) + F_i(\bar{e}_{i'}) < \sum_i F_i(e_i^{**}) \tag{2.3a}$$

and

$$\sum_{i \neq i'} h_{ij}e_i^{**} + h_{i'j}\bar{e}_i < \sum_i h_{ij}e_i^{**}. \tag{2.3b}$$

By (2.3b) the vector $(e_1^{**},..., \bar{e}_{i'},..., e_n^{**})$ satisfies $EH \leqslant Q^*$ and by (2.3a) E^{**} does not minimize $\sum_i F_i(e_i)$.

3. MARKETS IN LICENSES

We can now proceed to the construction of markets which, in equilibrium, lead to emission rates which satisfy the conditions of Theorem 2.1. A set of licenses are defined, such that the possession of licenses confers the right to carry out a certain average rate of emission.

Consider the function

$$\Lambda(H_i, L_i),$$

where H_i is the i-th row of the matrix H and $L_i = (l_{i1}, ..., l_{ik})$. We define l_{ik} as the number of licenses of type k held by firm i. This function defines the right to emit which is generated by holding a portfolio of licenses L_i. Then firm i can maximize profits by minimizing direct emission costs plus the cost of purchasing licenses, subject to the constraint that emissions not exceed $\Lambda(H_i, L_i)$. We assume throughout that some initial allocation of licenses l_{ik}^0, is made. Then the firm's problem is to minimize

$$F_i(e_i) + \sum_k p_k(l_{ik} - l_{ik}^0)$$

subject to $e_i \leqslant \Lambda(H_i, L_i)$.

A market equilibrium will exist if there exist nonnegative prices P^* such that when e_i^*, L_i solve the firm's minimization problem for p_k^*, the following market clearing conditions hold:

$$\sum_i (l_{ik}^* - l_{ik}^0) \leqslant 0, \qquad \sum_k p_k^* \left[\sum_i (l_{ik}^* - l_{ik}^0)\right] = 0. \tag{3.1a}$$

That is, there is some set of prices of licenses such that when each firm minimizes the sum of the cost of reducing emissions and the net cost of buying licenses, excess demand for licenses is nonpositive, and excess supply of a license drives its price to zero.

The market equilibrium is efficient if e_i^* represents equilibrium emissions and in any equilibrium

$$\sum_i F_i(e_i^*) = \sum_i F_i(e_i^{**}). \tag{3.1b}$$

Note that when all licenses are allocated to firms, (3.1a) implies that any expenditure on licenses by one firm is a revenue to another firm. Therefore total expenditure among all firms, associated with the control of pollution, just equals the total cost of emission control. That is,

$$\sum_i \left[F_i(e_i^*) + \sum_k p_k(l_{ik}^* - l_{ik}^0)\right] = \sum_i F_i(e_i^*) + \sum_k p_k \left[\sum_i (l_{ik}^* - l_{ik}^0)\right]$$
$$= \sum_i F_i(e_i^*).$$

These three properties do not exhaust the set of desirable properties of a market system. It might be that an equilibrium exists, or is efficient, only under strong conditions on the initial allocations of licenses which can be adopted. The more variation which is possible in the choice of initial allocations, the more freedom the management agency will have to pursue such goals as equity of the treatment, subsidization of "deserving" industries, and so on. We begin by defining a licensing system which has an efficient equilibrium for all distributions of a fixed total of licenses.

Analogously to the distinction between ambient standards and emission standards, we must differentiate between emission licenses and pollution licenses. Emission licenses are perhaps the most natural to think of trading, but there are great problems in using them when quality at many locations is a matter of concern. In particular, it is not possible to allow the licenses to be traded on a one-for-one basis [12].

Suppose there are two sources of pollution and one monitoring point, that each source is assigned licenses which allow it to emit 5 units of pollutant, and that $h_{11} = 1$ and $h_{21} = 2$. Under these circumstances, there will be $5 \cdot 1 + 5 \cdot 2 = 15$ units of pollution at the monitoring point. The marginal rate of substitution between emissions at source 2 and emissions at source 1 which keeps air quality constant is 2. If licenses are exchanged on a one-for-one basis, the transfer of one license from firm 1 to firm 2 will result in air quality being degraded to 16 units of pollution. If there is a second monitoring point, and $h_{22}/h_{12} \neq 2$, the marginal rate of substitution between emissions at sources 1 and 2 will change, depending on which monitoring point imposes the *operative* constraint on emissions. By defining rights to *cause* pollution at each of the monitoring points, we can avoid these problems completely, although they can be resolved with emission licenses if certain restrictions on trades are observed.

3.1. *The Market in Licenses to Pollute*

In this section we establish the existence and efficiency of equilibrium in a system of transferable licenses to pollute. Let l_{ij} represent the quantity of licenses allowing pollution at point j held by firm i, and let

$$L_i = (l_{i1}, \ldots, l_{im})$$

be the "portfolio" of licenses held by firm i. The licensing function can have the form

$$\Lambda(H_i, L_i) = \min_j \frac{l_{ij}}{h_{ij}},$$

which implies that each firm faces the constraints

$$h_{ij} e_i \leqslant l_{ij} \qquad j = 1, \ldots, m.$$

That is, the relevant element of the diffusion matrix is taken to be a correct predictor of the amount which an average rate of emission at point i contributes to pollutant concentration at point j. Each firm is allowed to have an average rate of emission which produces no more pollution at any point than the amount which the firm is licensed to cause at that point. The firm will minimize $F_i(e_i) + \sum_j p_j(l_{ij} - l_{ij}^0)$ subject to the licensing constraint.

In the theorems which follow we use the convention that l_j is a scalar, a total number of licenses allowing pollution at point j. Thus $\sum_i l_{ij} = l_j$. When $L_i = (l_{i1},..., l_{im})$ and $L = (l_1,..., l_m)$, $\sum_i L_i = L$.

The strategy of proof is to define a market equilibrium relative to an initial allocation of licenses and to derive necessary and sufficient conditions for its existence. A subsidiary construction, called a "license-constrained joint cost minimum," is defined and shown to exist. It generates a second set of necessary and sufficient conditions. It is shown that the emission vector and shadow prices which satisfy the conditions of a license-constrained joint cost minimum for given totals of licenses also satisfy the conditions of competitive equilibrium relative to any initial allocation of licenses in which the given totals are completely distributed among firms. An equilibrium license portfolio for each firm is constructed, and shadow prices on each firm's licensing constraints are identified. To prove that a competitive equilibrium achieves the joint cost minimum defined in Section 2, we show that when license totals equal desired air qualities any emission vector and price vector which satisfy the equilibrium conditions also satisfy the conditions for efficiency. In the course of the proof the efficient emission vector is identified as the equilibrium emission vector and shadow prices on the air quality constraints in the overall joint cost minimum are identified as the prices of licenses. The equilibrium license portfolio has each element just equal to the pollution caused by the efficient rate of emission for the corresponding firm. The proof itself is rigorous and abstract.

DEFINITION. A market equilibrium is an $n + 2$ tuple of vectors $L_i^* \geqslant 0$, $E^* \geqslant 0$, and $P^* \geqslant 0$ such that L_i^* and E^* minimize

$$F_i(e_i) + \sum_j p_j^*(l_{ij} - l_{ij}^0)$$

subject to $l_{ij} - h_{ij}e_i \geqslant 0, j = 1,..., m$, for all i and which also satisfy the market clearing conditions

$$\sum_i (l_{ij}^* - l_{ij}^0) \leqslant 0, \qquad \sum_j p_j^* \left[\sum_i (l_{ij}^* - l_{ij}^0)\right] = 0. \qquad (3.1a)$$

LEMMA 3.1. *A market equilibrium exists if and only if there exist vectors*

$$(u_{i1}^*, \ldots, u_{im}^*) \geqslant 0 \qquad i = 1, \ldots, n,$$
$$(p_1^*, \ldots, p_m^*) \geqslant 0$$

such that

$$F_i'(e_i^*) + \sum_j u_{ij}^* h_{ij} \geqslant 0, \qquad e_i^* \left[F_i'(e_i^*) + \sum_j u_{ij}^* h_{ij}\right] = 0, \quad (3.2a)$$

$$p_j^* - u_{ij}^* \geqslant 0, \qquad \sum_j l_{ij}^* [p_j^* - u_{ij}^*] = 0, \quad (3.2b)$$

$$l_{ij}^* - h_{ij} e_i^* \geqslant 0, \qquad \sum_j u_{ij}^* [l_{ij}^* - h_{ij} e_i^*] = 0, \quad (3.2c)$$

for all i and

$$\sum_i (l_{ij}^* - l_{ij}^0) \leqslant 0, \qquad \sum_j p_j^* \left[\sum_i (l_{ij}^* - l_{ij}^0)\right] = 0. \quad (3.2d)$$

Proof. First we characterize the vectors L_i^* and e_i^* which minimize cost for the firm. Minimizing a function is equivalent to maximizing its negative; and the negative of a convex function is concave. Therefore, we can state the problem of the firm as one of maximizing the concave function

$$-F_i(e_i) - \sum_j p_j^* (l_{ij} - l_{ij}^0).$$

Form the Lagrangean

$$\phi_i(l_{i1}, \ldots, l_{im}, e_i, u_{i1}, \ldots, u_{im})$$
$$= -F_i(e_i) - \sum_j p_j^* (l_{ij} - l_{ij}^0) + \sum_j u_{ij} (l_{ij} - h_{ij} e_i).$$

From the Kuhn–Tucker theorem the following conditions are necessary and sufficient for the constrained maximum; where $\phi_i(l_{i1}^*, \ldots, l_{in}^*, e_i^*, u_{i1}^*, \ldots, u_{in}^*) = \phi_i^*$.

$$\partial \phi_i^* / \partial e_i \leqslant 0, \qquad e_i^* \cdot (\partial \phi_i^* / \partial e_i) = 0,$$
$$\partial \phi_i^* / \partial l_{ij} \leqslant 0, \qquad \sum_j l_{ij}^* \cdot (\partial \phi_i^* / \partial l_{ij}) = 0,$$
$$\partial \phi_i^* / \partial u_{ij} \geqslant 0, \qquad \sum_j u_{ij}^* \cdot (\partial \phi_i^* / \partial u_{ij}) = 0.$$

Performing the indicated differentiation gives 3.2a to 3.2c which must be satisfied for all i. Equation (3.2d) repeats the market clearing condition.

DEFINITION. A license-constrained joint cost minimum is a vector E^{**} which minimizes

$$\sum_i F_i(e_i)$$

subject to $EH \leqslant L^0$ and $E \geqslant 0$.

In making this definition we assume that some arbitrary vector of licenses L^0 is issued by the management agency. We must assume that the set $\{E \mid EH \leqslant L^0 \text{ and } E \geqslant 0\}$ is not empty. Then the same argument used in Section 2 to establish the existence of a joint cost minimum will establish the existence of a license-constrained minimum. We now can use the following lemma to prove existence of an equilibrium on the pollution license market.

LEMMA 3.2. *An emission vector E^{**} is a license-constrained joint cost minimum if and only if there exists a vector $(u_1^{**},..., u_m^{**}) \geqslant 0$ such that*

$$F_i'(e_i^{**}) + \sum_j u_j^{**}h_{ij} \geqslant 0, \qquad \sum_i e_i^{**}\left[F_i'(e_i^{**}) + \sum_j u_j^{**}h_{ij}\right] = 0, \quad (3.3a)$$

$$l_j^0 - \sum_i h_{ij}e_i^{**} \geqslant 0, \qquad \sum_j u_j^{**}\left[l_j^0 - \sum_i h_{ij}e_i^{**}\right] = 0. \quad (3.3b)$$

Proof. The proof is as in Lemma 3.1.

The market equilibrium will exist for any distribution of licenses such that $l_{ij}^0 \geqslant 0$ and $\sum_i l_{ij}^0 = l_j^0$.

THEOREM 3.1. *A market equilibrium of the pollution license system exists for $\sum_i l_{ij}^0 = l_j^0$.*

Proof. We proceed constructively by using (3.3a) and (3.3b) to show that $e_i^* = e_i^{**}$, $l_{ij}^* = h_{ij}e_i^{**}$, $p_j^* = u_j^{**}$ and $u_{ij}^* = u_j^{**}$ for all i satisfy (3.2a)–(3.2d).

Equation (3.2a). Since $F_i'(e_i^{**}) + \sum_j u_j^{**}h_{ij} \geqslant 0$ for all i, and $e_i^{**} \geqslant 0$, it follows from

$$\sum_i e_i^{**}\left[F_i'(e_i^{**}) + \sum_j u_j^{**}h_{ij}\right] = 0$$

that

$$e_i^{**}\left[F_i'(e_i^{**}) + \sum_j u_j^{**}h_{ij}\right] = 0$$

for all i. Therefore, e_i^{**} and u_j^{**} satisfy 3.2a for all i.

Equation (3.2b). If $p_j^* = u_j^{**}$ and $u_{ij}^* = u_j^{**}$, $p_j^* - u_{ij}^* = 0$ for all i and j, and (3.2b) is satisfied by any l_{ij}^*.

Equation (3.2c). If $l_{ij}^* = h_{ij}e_i^{**}$ for all i and j, clearly l_{ij}^* and e_i^{**} satisfy (3.2c) for any u_{ij}^*, and in particular for u_j^{**}.

Equation (3.2d). Let $\sum_i l_{ij}^0 = l_j^0$ and $l_{ij}^* = h_{ij}e_i^{**}$. Then, (3.3b) gives by substitution

$$\sum_i l_{ij}^0 - \sum_i l_{ij}^* \geqslant 0 \qquad \text{and} \qquad \sum_j u_j^{**}\left[\sum_i l_{ij}^0 - \sum_i l_{ij}^*\right] = 0.$$

Therefore, $p_j^* = u_j^{**}$ and l_{ij}^* satisfy (3.2d).

Thus we conclude that for any choice of license totals which imply a feasible air quality vector, a market equilibrium exists. If we choose the license totals correctly, we can show that any market equilibrium is a joint cost minimum. The joint cost minimum was defined in Section 2 as a vector E^{**} which minimizes $\sum_i F_i(e_i)$ subject to $EH \leqslant Q^*$ and $E \geqslant 0$. First we prove that any emission vector which results from a market equilibrium with $\sum_i l_{ij}^0 = l_j^0$ minimizes $\sum_i F_i(e_i)$ subject to $EH \leqslant L^0$ and $E \geqslant 0$.

THEOREM 3.2. *Any emission vector which satisfies the conditions of a market equilibrium with* $\sum_i L_i^0 = L^0$ *is a license-constrained joint cost minimum.*

Proof. We show that any e_i^* which satisfies (3.2a)–(3.2d) satisfies (3.3a) and (3.3b) with $u_j^{**} = p_j^*$.

Equation (3.3a). By (3.2b), either $u_{ij}^* = p_j^*$ or $l_{ij}^* = 0$. By (3.2c), $l_{ij}^* \geqslant h_{ij}e_i^*$; therefore, whenever $p_j^* \neq u_{ij}^*$, $l_{ij}^* = 0$, and it follows that $e_i^* = 0$, or $h_{ij} = 0$. Whenever $h_{ij} = 0$, $p_j^* h_{ij} = u_{ij}^* h_{ij} = 0$. Therefore, $e_i^*[F_i'(e_i^*) + \sum_j p_j^* h_{ij}] = 0$ holds whether or not $p_j^* = u_{ij}^*$.

Since $u_{ij}^* \leqslant p_j^*$, $\sum_j u_{ij}^* h_{ij} \leqslant \sum_j p_j^* h_{ij}$ and $F_i'(e_i^*) + \sum_j u_{ij}^* h_{ij} \geqslant 0$ imply $F_i'(e_i^*) + \sum_j p_j^* h_{ij} \geqslant 0$. Therefore, e_i^* and p_j^* satisfy the inequality in (3.3a).

Equation (3.3b). Since $\sum_i l_{ij}^0 = l_j^0$, (3.2d) implies that $l_j^0 = \sum_i l_{ij}^0 \geqslant \sum_i l_{ij}^*$. Since, by (3.2c), $l_{ij}^* - h_{ij}e_i^* \geqslant 0$, $\sum_j l_{ij}^* - \sum_i h_{ij}e_i^* \geqslant 0$. If $l_j^0 \geqslant \sum_i l_{ij}^*$, then it must be true that $l_j^0 - \sum_i h_{ij}e_i^* \geqslant 0$ and the inequality in (3.3b) is satisfied by e_i^*.

Substitute l_j^0 for $\sum_i l_{ij}^0$ in (3.2d) giving $\sum_j p_j^*[l_j^0 - \sum_i l_{ij}^*] = 0$. If $l_{ij}^* = h_{ij}e_i^*$ for all i and all j, clearly $\sum_j p_j^*[l_j^0 - \sum_i h_{ij}e_i^*] = 0$. Assume that $l_{ij}^* - h_{ij}e_i^* > 0$ for some i and j. Then, by (3.2c), $u_{ij}^* = 0$. If $p_j^* \neq 0$, (3.2b) implies that $l_{ij}^* = 0$ for that i and j, and since $l_{ij}^* > h_{ij}e_i^*$ for that i and j, e_i^* must be negative. Since this is impossible, we must have either $l_{ij}^* = h_{ij}e_i^*$ for all i or $p_j^* = 0$; and the alternative holds for each j. Therefore,

$$p_j^* \left[\sum_i h_{ij}e_i^* - l_j^0\right] = 0,$$

and p_j^*, e_i^* satisfy (3.3b).

Thus, if we take the totals of each type of license distributed to firms we will find that firms exchange licenses so as to minimize joint total cost subject to the constraint that concentrations of pollutants at each monitoring point be no greater than the total of licenses issued for that point. The following corollary is immediate.

COROLLARY. *If $L^0 = Q^*$, the equilibrium emission vector is a joint cost minimum.*

Proof. If $L^0 = Q^*$, by Theorem 3.2 an equilibrium emission vector minimizes $\sum_i F_i(e_i)$ subject to $EH \leqslant Q^*$ and $E \geqslant 0$.

Theorem 3.1 can now be restated as "an efficient emission vector can be achieved as a competitive equilibrium" and 3.2 as "any competitive equilibrium with appropriate license distribution achieves an efficient vector." We can also prove an interesting theorem on the initial allocation of licenses.

THEOREM 3.3. *If $l_{ij}^0 \geqslant 0$ and $\sum_i l_{ij}^0 = q_j^*$, then E^*, P^*, and L_i^* are independent of L_i^0.*

Proof. Equations (3.2a)–(3.2c) depend in no way on L_i^0. In (3.2d) L_i^0 appears, but only in the form of the sum $\sum_i L_i^0$.

This result is somewhat unusual, in that the particular equilibrium achieved in a system usually depends on the initial allocations. The reason that this system is independent of the initial allocation is that the firm's behavior is independent of its asset position. Any redistribution which preserves totals of each type of license does not change the equilibrium. A graphical depiction of the equilibrium of the firm when a system of pollution licenses is imposed reveals the independence of initial allocations. The equilibrium is depicted in Fig. 1.

In the course of proving Theorem 3.1 it was noted that $p_j^*(h_{ij}e_i^* - l_{ij}^*) = 0$, so that $\sum_j p_j^*(h_{ij}e_i^* - l_{ij}^*) = 0$. Any emission level chosen by the firm implies that the firm purchases certain quantities of licenses, so that we can associate with any emission rate a cost equal to $\sum_j p_j^* h_{ij}e_i^*$. The minimization of cost (of emission control plus net purchases of licenses) can then be represented as the minimization of the sum $F_i(e_i) + \sum_j h_{ij} p_j^* e_i$. The emission rate e_i^* in Fig. 1 is the minimizer of this sum. Theorem 3.1 states that there exist prices which clear markets for licenses when each firm chooses license holdings, and emission rates e_i^*, to minimize cost.

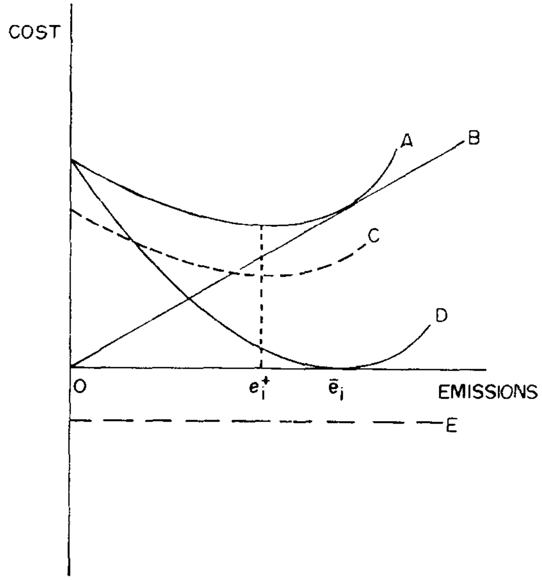

FIG. 1. A—$F_i(e_i) + \sum_j p_j h_{ij} e_i$; B—$\sum_j p_j h_{ij} e_i$; C—$F_i(e_i) + \sum_j p_j h_{ij} e_i - \sum_j p_j l_{ij}^0$; D—$F_i(e_i)$; E—$\sum_j p_j l_{ij}^0$.

The initial allocation of licenses is equivalent to a lump sum subsidy, and is independent of emission level. Therefore, this subsidy can be represented as a horizontal line, $\sum_j p_j^* l_{ij}^0$, in Fig. 1. The curve $F_i(e_i) + \sum_j p_j^* h_{ij} e_i - \sum_j p_j^* l_{ij}^0$ is the net cost function which represents the actual cost of emission control and licenses. Note that e_i^* is independent of the size of the subsidy. Because of this result, the management agency can distribute licenses as it pleases. Considerations of equity, of administrative convenience, or of political expediency can determine the allocation. The same efficient equilibrium will be achieved.

It should, however, be noted that in assuming the convexity of $F_i(e_i)$ we impose certain conditions relating to nonnegative profits. Let $\bar{\pi}_i$ be the (maximal) profit earned before the introduction of a licensing system, and let $\tilde{\pi}_i(e_i)$ be the profit earned when emissions are set at rate e_i. Then by (1.1) we have $\tilde{\pi}_i(e_i) = \bar{\pi}_i - F_i(e_i)$. In the long run the firm will only stay in business if $\tilde{\pi}_i(e_i) \geqslant 0$. In this case the cost function will have the form $F_i(e_i) = \min(\tilde{\pi}_i - \tilde{\pi}_i(e_i), \tilde{\pi}_i)$. An upper bound, equal to $\tilde{\pi}_i$, is placed on costs incurred by the firm. This upper bound destroys the convexity of the cost function unless $F_i(0) \leqslant \tilde{\pi}_i$. Such an assumption is implicit in the assumption that $F_i(e_i)$ is convex.

It would appear that the need to purchase licenses imposes a cost on the firm additional to the cost of emission control $F_i(e_i)$. Even though this cost sums to zero for all firms taken together, it may be positive for some individual firms and negative for others. Fortunately, we can prove the following theorem, namely, that if $F_i(0) \leqslant \tilde{\pi}_i$, then even if a firm is allocated no licences initially (i.e., $l_{ij}^0 = 0$ for some i and all j), it can still earn nonnegative profits at any levels of emissions and license holding.

THEOREM 3.4. *If* $F_i(0) \leqslant \tilde{\pi}_i$, $F_i(e_i^*) + \sum_j p_j^* l_{ij}^* \leqslant \tilde{\pi}_i$.

Proof. We have proved that

$$F_i(e_i^*) + \sum_j p_j^* l_{ij}^* = F_i(e_i^*) + \sum_j p_j^* h_{ij} e_i^*.$$

If $F_i(e_i^*) + \sum_j p_j^* h_{ij} e_i^* > \tilde{\pi}_i$, then $F_i(e_i^*) + \sum_j p_j^* h_{ij} e_i^* > F_i(0)$. But $e_i = 0$ and $l_{ij} = 0$ satisfy $e_i \leqslant \Lambda(H_i, L_i)$, so that e_i^* does not minimize cost subject to the licensing constraint. This contradiction establishes the theorem.

This demonstration completes the discussion of pollution licenses. We began by showing that for any vector of licenses L^0 which implies feasible concentration levels at each monitoring point there exists a competitive equilibrium in the license market. We then showed that the concentrations which result from the equilibrium will be less than or equal to the levels permitted by the vector of licenses and that joint total costs are minimized subject to this constraint. Finally we showed that when $L^0 = Q^*$, the problem of achieving desired air quality standards at minimum cost is solved by the market in pollution licenses.

The major generalization provided by this theorem is that it establishes the possibility of achieving environmental goals at a *number* of geographic points while maintaining the advantages of a market system. Thus one important objection to the use of economic incentives, that they could lead to change in the pattern of emissions such that although air quality improvements at one point are achieved, it is at the expense of deteriorating air quality elsewhere, is laid to rest. Moreover, we discovered that the fixed totals could be allocated arbitrarily among firms.

Overall convexity and the possibility, for each firm, of absorbing all costs of abatement in profits were assumptions necessary for the operation of the system when no information on cost functions is available.

We turn now to an alternative licensing system. It will turn out that this system of emission licenses is interesting because it provides a means of linking up the proposal to issue transferable licenses with other proposals for achieving efficient solutions in a decentralized manner.

3.2. *The Market in Emission Licenses*

The effluent charge is a tax which a firm must pay on each unit of pollutant which it emits into a water course. A corresponding charge for air pollution control might be called an emission charge. In order to calculate a charge which will lead to efficiency in air pollution control, the manager must solve in advance the overall cost-minimization problem. It is not difficult to show that the correct tax on emission by firm i is equal to the shadow price on its emissions determined by the minimization of joint total cost. The tax is $\sum_j u_j^* h_{ij}$, where u_j^* is the value of the Lagrange multiplier on the j-th quality constraint evaluated at the optimum. But in order to calculate such a tax the manager must know the cost functions of each firm. It is, of course, possible to obtain that information in an iterative process by varying the tax. This is a cumbersome and politically unattractive procedure, and it has been shown by Marglin [10] that the information transferred to the regulatory authority by such a procedure is as great as the information needed to set quantity standards for each firm. That is, whenever it is possible to calculate the correct tax it is possible to achieve E^{**} in the initial allocation.

A licensing scheme does not require such prior or iterative gathering of information. The market makes the necessary calculations independently in the course of reaching equilibrium. For this reason we are led to consider licensing schemes as superior to taxation. The natural correlate of emission charges is a system of emission licenses.

An emission license confers on the firm holding it the right to emit pollutants at a certain rate. It is not always desirable to allow such rights to be transferred on a one-for-one basis: the desirable rule governing exchange of emission rights is that a firm may be allowed to emit up to a level which causes pollution equal to that which would be caused if each firm from which it obtained rights emitted to the maximum extent permitted by the rights which it has given up. We must differentiate rights to emit by the location at which they permit emissions to take place. Then l_k, $k = 1,..., n$, is a quantity of licenses to emit at location k. It is sufficient to allow k to run over the set of firms I since each firm is in a fixed location. Let l_{ik} represent the quantity of licenses allowing emissions at location k held by firm i.

If the exchange of such licenses between polluters at different locations is to be permitted, some rule must be stated regarding the right to emit which a license to emit at location k confers on a firm at location i. Consider a firm i which emits at a rate $e_i = h_{kj}l_{ik}/h_{ij}$. Then the pollution which firm i causes at point j is precisely the pollution which firm k would cause if it emitted at the rate $e_k = l_{ik}$, since $h_{ij}e_i = h_{kj}l_{ik} = h_{kj}e_k$. The licensing function can have the form

$$\Lambda(H_i, L_i) = \min_j \left(\sum_k h_{kj}l_{ik}/h_{ij}\right),$$

which implies that each firm faces the constraints

$$h_{ij}e_i \leqslant \sum_k h_{kj}l_{ik} \qquad j = 1,..., m,$$

and will minimize

$$F_i(e_i) + \sum_k p_k(l_{ik} - l_{ik}^0)$$

subject to those constraints.

A restriction on the initial allocation of licenses is needed if the market equilibrium with emission licenses is to be efficient. It is that $\sum_i l_{ik}^0 \geqslant 0$ and $\sum_k h_{kj} \sum_i l_{ik}^0 = q_j{}^*$ for all j. Note that this assumption is equivalent to the assumption that there exists a nonnegative emission vector E^0 such that

$$E^0 \cdot H = Q^*.$$

This is quite a strong condition, since even if the matrix H is of full rank, for arbitrary semipositive H and Q^* the equations

$$E \cdot H = Q^*$$

will not in general have nonnegative solutions.

If $\sum_k h_{kj}l_k{}^0 < q_j{}^*$ for some j it may not be possible to achieve the minimum of joint total cost without prior knowledge of cost functions. Suppose that the joint cost minimum vector E^{**} satisfies

$$\sum_i h_{ij}e_i^{**} = q_j{}^*$$

for some j, and that $\sum_k h_{kj}l_k{}^0 < q_j{}^*$ for the same j. Then since the equilibrium emission vector E^* satisfies

$$\sum_i h_{ij}e_i{}^* \leqslant \sum_k h_{kj}l_k{}^0,$$

it follows that

$$\sum_i h_{ij} e_i^* < \sum_i h_{ij} e_i^{**}$$

and $e_i^* \neq e_i^{**}$, so that the market equilibrium is inefficient.

DEFINITION. A market equilibrium in emission licenses is an $n + 2$-tuple of vectors $L_i^* \geqslant 0$, $E^* \geqslant 0$ and $P^* \geqslant 0$ such that L_i^* and E^* minimizes

$$F_i(e_i) + \sum_k p_k^*(l_{ik} - l_{ik}^0)$$

subject to

$$\sum_k h_{kj} l_{ik} - h_{ij} e_i \geqslant 0 \qquad j = 1,..., m$$

and

$$e_i \geqslant 0; \qquad l_{ik} \geqslant 0$$

for all i and which also satisfy the market clearing conditions

$$\sum_i (l_{ik}^* - l_{ik}^0) \leqslant 0, \qquad \sum_k p_k^* \left[\sum_i (l_{ik}^* - l_{ik}^0)\right] = 0.$$

LEMMA 3.3. *A market equilibrium exists if there exist vectors*

$$\begin{aligned} &(u_{i1}^*,..., u_{im}^*) \geqslant 0 \qquad i = 1,..., n, \\ &(p_1^*,..., p_n^*) \geqslant 0, \end{aligned} \tag{3.4}$$

such that

$$F_i'(e_i^*) + \sum_j u_{ij}^* h_{ij} \geqslant 0, \qquad e_i^* \left[F_i'(e_i^*) + \sum_j u_{ij}^* h_{ij}\right] = 0, \tag{3.5a}$$

$$p_k^* - \sum_j u_{ij}^* h_{kj} \geqslant 0, \qquad \sum_k \left[l_{ik}^* \left(p_k^* - \sum_j u_{ij}^* h_{kj}\right)\right] = 0, \tag{3.5b}$$

$$\sum_k h_{kj} l_{ik}^* - h_{ij} e_i^* \geqslant 0, \qquad \sum_j \left[u_{ij}^* \left(\sum_k h_{kj} l_{ik}^* - h_{ij} e_i^*\right)\right] = 0, \tag{3.5c}$$

for all i and

$$\sum_i (l_{ik}^* - l_{ik}^0) \leqslant 0, \qquad \sum_k p_k^* \left[\sum_i (l_{ik}^* - l_{ik}^0)\right] = 0. \tag{3.5d}$$

Proof. The proof is as in Lemma 3.1.

THEOREM 3.5. *A market equilibrium in emission licenses exists.*

Proof. In Theorem 2.2 it was shown that an emission vector minimizing joint total costs subject to the air quality constraints exists, and that in consequence E^{**} and U^{**} satisfying (2.1) and (2.2) exist. Let licenses be issued initially so that

$$\sum_k h_{kj} \sum_i l_{ik}^0 = q_j^*$$

for all j. Then we show that E^{**} is an emission vector and U^{**} a price vector satisfying (3.5a)–(3.5d).

We begin by proving the following proposition:

P.1. *If* $\sum_k h_{kj} l_k^0 \geqslant \sum_i h_{ij} e_i^{**}$, *then there exist* l_{ik}^* *such that* $\sum_i l_{ik}^* \leqslant l_k^0$, $l_{ik}^* \geqslant 0$, *and* $\sum_k h_{kj} l_{ik}^* \geqslant h_{ij} e_i^{**}$ *for all i and k. Letting* $L_i = (l_{i1}, \ldots, l_{in})$ *and* H_i *be the i-th row of the matrix H we may write the inequalities which must have a nonnegative solution in matrix form as*

$$(L_1^*, \ldots, L_n^*) \begin{bmatrix} -H & & I \\ & \ddots & \vdots \\ & & -H I \end{bmatrix} \leqslant (-H_1 e_1^{**}, \ldots, -H_n e_n^{**}, L^0).$$

It is a theorem [3] *that either these inequalities or the following inequalities have a nonnegative solution*:

$$\begin{bmatrix} -H & & I \\ & \ddots & \vdots \\ & & -H I \end{bmatrix} \begin{bmatrix} X_1 \\ \vdots \\ X_n \\ X_{n+1} \end{bmatrix} \geqslant 0, \tag{1}$$

$$(-H_1 e_1^{**}, \ldots, -H_n e_n^{**}, L) \begin{bmatrix} X_1 \\ \vdots \\ X_n \\ X_{n+1} \end{bmatrix} < 0. \tag{2}$$

We can write (1) as

$$-\sum_{j=1}^{m} h_{ij} x_{1j} + x_{n+1i} \geqslant 0 \qquad (i = 1, \ldots, n)$$

$$\vdots$$

$$-\sum_{j=1}^{m} h_{ij} x_{nj} + x_{n+1i} \geqslant 0 \qquad (i = 1, \ldots, n)$$

and (2) as

$$-\sum_{i=1}^{n}\sum_{j=1}^{m} h_{ij}e_i{}^*x_{ij} + \sum_i l_i{}^0 x_{n+1i} < 0.$$

We assume that there do exist nonnegative solutions denoted with superscript 0's to (1) and (2). Let us multiply each line of (1) through by $l_i{}^0$ and sum the result over i, giving

$$-\sum_{j=1}^{m}\sum_{i=1}^{n} h_{ij}l_i{}^0 x_{kj} + \sum_i l_i{}^0 x_{n+1i} \geqslant 0$$

for all k.

Comparing this inequality with (2) we find

$$-\sum_{i=1}^{n}\sum_{j=1}^{m} h_{ij}e_i{}^*x_{ij} < -\sum_{i=1}^{n}\sum_{j=1}^{m} h_{ij}l_i{}^0 x_{kj}\,.$$

Since $\sum_i h_{ij}e_i{}^* \leqslant \sum_i h_{ij}l_i{}^0$ by hypothesis,

$$-\sum_j x_{kj}^0 \sum_i h_{ij}l_i{}^0 \leqslant -\sum_j x_{kj}^0 \sum_i h_{ij}e_i{}^*,$$

and

$$-\sum_{i=1}^{n}\sum_{j=1}^{m} h_{ij}e_i{}^*x_{ij}^0 < -\sum_{i=j}^{n}\sum_{j=1}^{m} h_{ij}e_i{}^*x_{kj}^0$$

for all k. We remove the minus signs and reverse the inequality, giving

$$\sum_i e_i{}^* \sum_j h_{ij}x_{ij}^0 > \sum_i e_i{}^* \sum_j h_{ij}x_{kj}^0$$

for all k. Therefore it must be true for some i that

$$\sum_j h_{ij}x_{ij}^0 > \sum_j h_{ij}x_{kj}^0$$

for all k. Therefore, it must be true for $k = i$, which implies

$$h_{i1}x_{i1}^0 + \cdots + h_{in}x_{in}^0 > h_{i1}x_{i1}^0 + \cdots + h_{in}x_{in}^0\,.$$

This contradiction establishes that there is no nonnegative solution to inequalities (1) and (2) and P.1 is proved.

We can now proceed line by line to show that E^{**} and U^{**} satisfying (2.1) and (2.2) also satisfy (3.5a)–(3.5d).

Equation (3.5a). From (2.1), e_i^{**} and u_j^{**} satisfy (3.5a) for all i.

Equation (3.5b). Let $p_k^* = \sum_j u_j^{**} h_{kj}$ and $u_{ij}^* = u_j^{**}$ for all i. Then they satisfy (3.5b) since $p_k^* - \sum_j u_{ij}^* h_{kj} = 0$ for all i and k.

Equation (3.5c). Let $\sum_k h_{kj} \sum_i l_{ik}^0 = q_j^*$. Then by (2.2),

$$0 \leqslant \sum_k h_{kj} \sum_i l_{ik}^0 - \sum_i h_{ij} e_i^{**},$$

and by P.1 there exist $l_{ik}^* \geqslant 0$ such that

$$\sum_i l_{ik}^0 \geqslant \sum_i l_{ik}^* \qquad \text{and} \qquad \sum_k h_{kj} l_{ik}^* - h_{ij} e_i^{**} \geqslant 0$$

for all i. If $>$ holds for some i and j,

$$q_j^* = \sum_k h_{kj} \sum_i l_{ik}^0 \geqslant \sum_k h_{kj} \sum_i l_{ik}^* > \sum_i h_{ij} e_i^{**},$$

and $u_j^{**} = 0$. Therefore, (3.5c) is satisfied with $u_{ij}^* = u_j^{**}$.

Equation (3.5d). If $\sum_i l_{ik}^0 > \sum_i l_{ik}^*$ for some k and $h_{kj} > 0$,

$$q_j^* = \sum_k h_{kj} \sum_i l_{ik}^0 > \sum_k h_{kj} \sum_i l_{ik}^* \geqslant \sum_i h_{ij} e_i^{**},$$

and $u_j^{**} = 0$ for all j. If $h_{kj} = 0$ for that k and some j, then for the corresponding j, $u_j^{**} h_{kj} = 0$. In either case $p_k^* = \sum_j u_j^{**} h_{kj} = 0$ and (3.5d) is satisfied.

We reverse the direction of inference to prove that if $\sum_i h_{ij} l_i^0 = q_j^*$, the competitive equilibrium emission vector is efficient. We assume in addition that the rank of H is m: this involves no significant loss of generality since any constraint matrix can be made to satisfy the condition by striking out redundant constraints. The operation of eliminating redundant constraints does not change the set Ψ of emission vectors which satisfy the constraints.

THEOREM 3.6. *If* $\sum_k h_{kj} l_k^0 = q_j^*$, E^* *minimizes* $\sum_i F_i(e_i)$ *subject to* $EH \leqslant Q^*$, $E \geqslant 0$.

Proof. First we note that in proving Theorem 3.4 we established that (3.5a)–(3.5d) are satisfied, for all i, by $u_{ij}^* = u_j^{**}$, and that the rank of H

equals m. Therefore, the matrix of partial derivatives of the licensing constraints for each firm also has rank m, and the multipliers on those constraints are unique [4]. Since the Kuhn–Tucker conditions are satisfied by identical multipliers for each firm, they are *only* satisfied by identical multipliers. Let u_j^{**} be equal to any of the u_{ij}^*, identical for all i. Then,

Equation (2.1). e_i^* and u_j^{**} as defined satisfy (2.1) whenever they satisfy (3.5a).

Equation (2.2). By (3.5d)

$$\sum_i l_{ik}^0 \geqslant \sum_i l_{ik}^*, \qquad \text{and} \qquad q_j^* = \sum_k h_{kj} \sum_i l_{ik}^0 \geqslant \sum_k h_{kj} \sum_i l_{ik}^*.$$

By (3.5c)

$$\sum_k h_{kj} \sum_i l_{ik}^* \geqslant \sum_i h_{ij} e_i^*.$$

Therefore $q_j^* - \sum_i h_{ij}e_i^* \geqslant 0$. If $q_j^* > \sum_i h_{ij}e_i^*$, either $\sum_i l_{ik}^0 > \sum_i l_{ik}^*$ for some k with $h_{kj} \neq 0$, or $\sum_k h_{kj} l_{ik}^* > h_{ij}e_i^*$ for some i and that j. If the latter, $u_{ij}^* = 0$ and $u_j^{**} = 0$. If the former, $p_k^* = 0$ and since by (3.5b) $p_k^* - \sum_j u_{ij}^* h_{kj} \geqslant 0$, $\sum_j u_{ij}^* h_{kj} = 0$ and $u_{ij}^* = 0$, so that $u_j^{**} = 0$. Therefore, $u_j^{**}(q_j^* - \sum_i h_{ij}e_i^*) = 0$ when $u_j^{**} = u_{ij}^*$.

This completes the proof that a competitive equilibrium, satisfying the conditions of joint cost minimization, exists in the market for emission licenses. An integral part of the proof was the assumption that the total of each type of license is determined by solving the *equations*

$$\sum_k h_{kj} l_k^0 = q_j^*.$$

If the management agency is restricted to assigning all licenses of type i which it issues to firm i, then its ability to redistribute costs will be severely limited by the necessity of choosing l_i^0 to satisfy the air quality constraints with equality if indeed such a solution exists in the problem at hand.

References

1. K. J. Arrow, The organization of economic activity: Issues pertinent to the choice of market versus non-market allocation, *in* "The Analysis and Evaluation of Public Expenditures: The PPB System," pp. 47–64, U.S. Congress, Joint Economic Committee, U.S. Government Printing Office, Washington, DC, 1969.
2. J. H. Dales, "Pollution, Property, and Prices," University of Toronto Press, Toronto, Canada, 1968.

3. D. Gale, "The Theory of Linear Economic Models," p. 47, McGraw-Hill, New York, 1960.
4. G. Hadley, "Nonlinear and Dynamic Programming," p. 190, Addison-Wesley Publ. Co., Inc., Reading, MA, 1964.
5. H. W. Herzog, The air diffusion model as an urban planning tool, *Socio-Econ. Plan. Sci.* 3 (1969), 329–349.
6. H. Jacoby and G. Schaumburg, "Administered Markets in Water Quality Control: A Proposal for the Delaware Estuary," unpublished.
7. S. Karlin, "Mathematical Methods and Theory in Games, Programming and Economics," Vol. I, pp. 203–204, Addison-Wesley Publ. Co., Inc., Reading Mass., 1962.
8. A. Kneese and B. Bower, "Managing Water Quality: Economics, Technology, Institutions," Chapter 2, Johns Hopkins Univ. Press, Baltimore, 1968.
9. S. Marglin, Objectives of water resources development: A general statement, *in* "Design of Water Resource Systems" (A. Maass *et al.*, Eds.), p. 23, Harvard Univ. Press, Cambridge, MA, 1962.
10. S. Marglin, Information in price and command systems of planning, *in* "Public Economics" (J. Margolis, Ed.), St. Martin's Press, New York, 1969.
11. D. O. Martin and J. A. Tikvart, A general atmospheric diffusion model for estimating the effects on air quality of one or more sources, APCA Paper No. 68-148, presented at the Annual Meeting of the Air Pollution Control Association, June 1968.
12. W. D. Montgomery, Artificial markets and the theory of games, Social Science Working Paper No. 8, California Inst. of Technol., March 1972.
13. P. Samuelson, "Foundations of Economic Analysis," pp. 34–35, Harvard Univ. Press, Cambridge, MA, 1966.
14. TRW Systems Group, "Regional Air Pollution Analysis, Phase I: Status Report," pp. 4.3–4.15, U.S. Department of Health, Education and Welfare and National Air Pollution Control Administration, Washington, DC, 1969.